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States as Morphisms

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Abstract Using elementary categorical methods, we survey recent results concerning *D*-posets (equivalently effect algebras) of fuzzy sets and the corresponding category *ID* in which states are morphisms. First, we analyze the canonical structures carried by the unit interval $I = [0,1]$ as the range of states and the impact of "states as morphisms" on the probability domains. Second, we analyze categories of various quantum and fuzzy structures and their relationships. Third, we describe some basic properties of *ID* and show that traditional probability domains such as fields of sets and bold algebras can be viewed as full subcategories of *ID* and probability measures on fields of sets and states on bold algebras become morphisms. Fourth, we discuss the categorical aspects of the transition from classical to fuzzy probability theory. We conclude with some remarks about generalized probability theory based on *ID*.

Keywords *D*-poset · State · Fuzzy probability theory · Probability domain · *ID*-poset · Algebraic quantum structure · Categorical methods · Epireflection

1 Introduction

Basic notions in probability theory are random events, observables (dual maps to random variables) and probability measures. To employ categorical methods, we redefine these basic notions in such a way that events become an object and both the observables and the probability measures become morphisms acting on such objects. To model properties of random events, in generalized probability theory many algebraic structures have been considered, see for example the well-known monograph [9] and the illuminating surveys [10, 28]. In the

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present paper we study random events mainly from the viewpoint of "states as morphisms". Undoubtedly, such approach has an impact on the "language" and the formalism, but we hope that the (elementary) categorical approach and methods provide a tool to put things into a perspective and to ask "good" questions. Final sections of the paper are devoted to various aspects of fuzzy (or operational) probability, see [3, 4, 15, 17, 19, 21, 25].

2 Range of states

In the classical, as well as in the generalized probability, a state is a mapping p from a probability domain into $I = [0, 1]$ preserving the structure of the domain only to some extent. In the classical case the domain is a σ-field of subsets and p preserves the order, but p is additive which means that $p(A \cup B) = p(A) + p(B)$ does not hold in general, e.g., $p(A \cup A) = p(A) \neq p(A) + p(A)$ for $p(A) > 0$. Similarly, in case of a Łukasiewicz tribe (a bold algebra) $\mathcal{X} \subseteq I^X$, $p(u \oplus v) = p(u) + p(v)$, $u, v \in \mathcal{X}$, does not hold in general.

Question 2.1 *Is there a category in which traditional probability domains* (*classical and fuzzy*) *are reorganized into objects and states are reorganized into structure preserving morphisms?*

Of course, we have in mind a category in which the reorganized traditional probability domains form distinguished subcategories. We claim that the answer is YES, it is the category *ID* of D-posets of fuzzy sets as objects and sequentially cotinuous D-homomorphisms as morphisms. The range of states plays a key role: I carrying a suitable structure cogenerates the category *ID*.

Observation 2.2 *If a state is a morphism*, *then its range* I *is an object*. *This of course means that first of all we have to study the particular structures of* I.

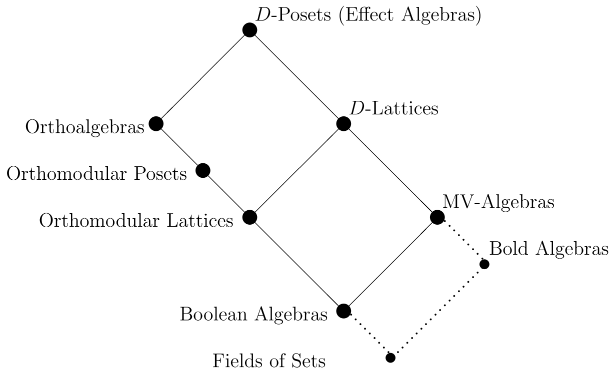
Observation 2.3 I *carries many natural structures*. *Some of them are fundamental when having in mind properties preserved by all states*: *order*, *the top and bottom elements*, *"difference"*, *convergence of sequences* (*indeed*, *the Lebesgue Dominated Convergence Theorem means that if a sequence of measurable* $[0, 1]$*-valued or* $\{0, 1\}$*-valued function pointwise converges*, *then the limit function is measurable and its integral is the limit of integrals of the elements of the sequence and hence*, *in particular*, *each probability measure is sequentially continuous*), $\dots$.

Observation 2.4 *Structures preserved by all states should characterize probability domains*.

Observation 2.5 *The structure of traditional probability domains and the structure of the objects of the new category should be "the same"*.

This leads to D-posets of fuzzy sets. As shown in Frič [12, 15–17, 19] and Papčo [24–26], *ID* is a suitable category in which basic notions of a probability theory having quantum character can be defined in a natural way. Generalized probability has been studied also in the realm of D-posets, see [7, 8, 14, 16, 22, 23]. In the next section we study D-posets in a broader context of algebraic quantum structures, cf. [9].

Fig. 1 Hasse diagram of structures



3 *D*-posets

A **difference poset** (a **D-poset**, in short) is a quintuple $(\mathcal{P}, \leq, \ominus, 0_{\mathcal{P}}, 1_{\mathcal{P}})$, where $\mathcal{P}$ is a partially ordered set with the least element $0_{\mathcal{P}}$ and the greatest one $1_{\mathcal{P}}$, $\ominus$ is a partial binary operation on $\mathcal{P}$, called a difference, such that $b \ominus a$ is defined if and only if $a \leq b$, and the following axioms are assumed

(D1) $a \ominus 0_{\mathcal{P}} = a$ for each $a \in \mathcal{P}$;
(D2) If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

A lattice ordered D-poset $(\mathcal{P}, \vee, \wedge, \ominus, 0_{\mathcal{P}}, 1_{\mathcal{P}})$ is called a **D-lattice**.

For any element a in a D-poset $\mathcal{P}$, the element $1_{\mathcal{P}} \ominus a$ is called the **orthosupplement** of a and is denoted by $a^{\perp}$. The unary operation $\perp$ is an involution and an order reversing operation on $\mathcal{P}$, i.e., $a^{\perp\perp} = a$, and, if $a \leqq b$ then $b^{\perp} \leqq a^{\perp}$.

In a D-poset, we can define a partial binary operation (orthosummation) $\oplus$ dual to the operation $\ominus$ as follows:

$$a \oplus b = (b^{\perp} \ominus a)^{\perp}, \quad \text{for } a \leqq b^{\perp}.$$

The orthosummation $\oplus$ is commutative, associative and $a \oplus a^{\perp} = 1_{\mathcal{P}}$.

We say that two elements a, b from a D-poset $\mathcal{P}$ are

(i) **orthogonal**, and write $a \perp b$, if $a \leqq b^{\perp}$;
(ii) **compatible**, and write $a \leftrightarrow b$, if there exist elements $c, d \in \mathcal{P}$ such that $d \leqq a \leqq c$, $d \leqq b \leqq c$ and $c \ominus a = b \ominus d$ (equivalently $c \ominus b = a \ominus d$).

If $\mathcal{P}$ is a D-lattice, then $a \leftrightarrow b$ if and only if $(a \vee b) \ominus a = b \ominus (a \wedge b)$.

D-posets generalize Boolean algebras cf. [29], orthomodular lattices and posets cf. [27], orthoalgebras cf. [11], as well as MV-algebras cf. [6]. The hierarchy of these structures can be schemed by the following Haase diagram (Fig. 1), where each vertice represents the corresponding algebraic structure and each edge represents a generalization. To the left are the so-called regular structures (recall that a structure is said to be regular if $a \leq a^{\perp}$ implies $a = 0$). According to this hierarchy, each Boolean algebra is an orthomodular lattice, each orthomodular lattice is an orthomodular poset and each orthomodular poset is an orthoalgebra. Further, each orthoalgebra is a D-poset, each orthomodular lattice is a D-lattice and each Boolean algebra is an MV-algebra. To the right, each MV-algebra is a D-lattice and each D-lattice is a D-poset. Note that fields of sets and bold algebras are special Boolean algebras and MV-algebras, respectively. Within the difference posets theory, a detailed description of these structures and their the relationships can be found in [7].

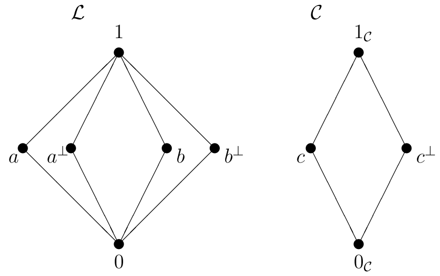


Fig. 2 Example 3.2

Let $\mathcal{P}$ and $\mathcal{T}$ be D-posets. A mapping $w: \mathcal{P} \to \mathcal{T}$ is called a **D-homomorphism** of $\mathcal{P}$ into $\mathcal{T}$ if the following conditions are satisfied

(DH1) $w(1_{\mathcal{P}}) = 1_{\mathcal{T}}$;
(DH2) If $a, b \in \mathcal{P},\ b \leq a$, then $w(a) \leq w(b)$;
(DH3) If $a, b \in \mathcal{P}, b \leq a$, then $w(b \ominus a) = w(b) \ominus w(a)$.

A D-homomorphism $w: \mathcal{P} \to \mathcal{T}$ is called a σ-D-homomorphism if, moreover, the following condition holds

(DH4) If $(a_n)_{n=1}^{\infty} \subseteq \mathcal{P},\ a_n \nearrow a,\ a \in \mathcal{P}$, (i.e., $a_n \leqq a_{n+1}$ for any $n \in N$ and $a = \bigvee_{n=1}^{\infty} a_n$), then $w(a_n) \nearrow w(a)$.

The following properties result directly from the definition of a D-homomorphism

(i) $w(0_{\mathcal{P}}) = 0_{\mathcal{T}}$;
(ii) $w(a^{\perp}) = (w(a))^{\perp}$ for any $a \in \mathcal{P}$;
(iii) If $a \perp b$ then $w(a) \perp w(b)$ and $w(a \oplus b) = w(a) \oplus w(b)$;
(iv) If $a \leftrightarrow b$ then $w(a) \leftrightarrow w(b)$.

Recall that an orthoalgebra [11] is a set $\mathcal{O}$ containing two special elements $0, 1$ and equipped with a partially defined binary operation $\oplus$ satisfying the following conditions for all $a, b, c \in \mathcal{O}$:

(O1) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$ (commutativity);
(O2) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$ (associativity);
(O3) For any $a \in \mathcal{O}$ there exists a unique $b \in \mathcal{O}$ such that $a \oplus b$ is defined and $a \oplus b = 1$ (orthosupplementation);
(O4) If $a \oplus a$ is defined, then $a = 0$ (consistency).

Theorem 3.1

(i) *Let $w: \mathcal{O}_1 \to \mathcal{O}_2$ be a D-homomorphism of an orthoalgebra $\mathcal{O}_1$ into an orthoalgebra $\mathcal{O}_2$. Then w is an orthoalgebra homomorphism.*
(ii) *Let $w: \mathcal{P}_1 \to \mathcal{P}_2$ be a D-homomorphism of an orthomodular poset $\mathcal{P}_1$ into an orthomodular poset $\mathcal{P}_2$. Then w is an orthomodular poset homomorphism.*
(iii) *Let $w: \mathcal{B}_1 \to \mathcal{B}_2$ be a D-homomorphism of a Boolean algebra $\mathcal{B}_1$ into a Boolean algebra $\mathcal{B}_2$. Then w is a Boolean homomorphism.*

Proof (i) This result follows immediately from the definition and the basic properties of the D-homomorphism.

(ii) It suffices to prove that D-homomorphism of orthomodular posets preserves the supremum of orthogonal elements. Let a and b be orthogonal elements from an orthomodular poset $\mathcal{P}$, i.e., $a \leqq b'$. Then

$$a \vee b = (a' \wedge b')' = (a' \ominus b)' = a \oplus b,$$

therefore,

$$w(a \vee b) = w(a \oplus b) = w(a) \oplus w(b) = w(a) \vee w(b).$$

(iii) It suffices to prove that D-homomorphism of Boolean algebras preserves the supremum of arbitrary two elements. We have

$$a \vee b = a \vee ((a \wedge b) \vee (a' \wedge b)) = a \vee (a' \wedge b),$$

and hence

$$w(a) \vee w(b) \leq w(a \vee b) = w(a) \vee w(a' \wedge b) \leq w(a) \vee w(b).$$ □

It is interesting to observe that if w is a D-homomorphism of an orthomodular lattice $\mathcal{P}$ into an orthomodular lattice $\mathcal{T}$, then, in general, w does not preserves the lattice structure.

Example 3.2 Let $\mathcal{L}$ be a 0-1-pasting of Boolean algebras $\mathcal{A} = \{0, 1, a, a^\perp\}$ and $\mathcal{B} = \{0, 1, b, b^\perp\}$. Let $\mathcal{C} = \{0_\mathcal{C}, 1_\mathcal{C}, c, c^\perp\}$. We define a mapping $w : \mathcal{L} \to \mathcal{C}$ such that $w(0) = 0_\mathcal{C}$, $w(1) = 1_\mathcal{C}$, $w(a) = w(b) = c$, $w(a^\perp) = w(b^\perp) = c^\perp$. The mapping w is a D-morphism where $w(a \vee b) = w(1) = 1_\mathcal{C}$ and $w(a) \vee w(b) = c$.

Denote D the category of D-posets and D-homomorphisms. It is known that D-posets and effect algebras are equivalent structures, in fact, they form isomorphic categories (cf. [8–10]). The same is true for their subcategories of D-posets of fuzzy sets and effect algebras of fuzzy sets (cf. [15, 26]).

Corollary 3.3

(i) *The category of orthoalgebras is isomorphic to the corresponding full subcategory of D.*
(ii) *The category of orthomodular posets is isomorphic to the corresponding full subcategory of D.*
(iii) *The category of Boolea algebras is isomorphic to the corresponding full subcategory of D.*

States and observables constitute fundamental notions of quantum probability theory on D-posets. They are defined as D-homomorphisms cf. [8].

A σ-D-homomorphism x of the σ-algebra $\mathcal{B}(R)$ of all Borel subsets of the real line R into a D-poset $\mathcal{T}$ is called an **observable** (on $\mathcal{T}$).

A σ-D-homomorphism s of a D-poset $\mathcal{P}$ into the unit interval $[0, 1]$ with the usual difference of real numbers is called a **state** (a probability measure) on $\mathcal{P}$.

4 *ID*-posets

Denote I the closed unit interval [0,1] carrying the usual linear order and the usual D-structure: $a \ominus b$ is defined whenever $b \le a$ and then $a \ominus b = a - b$. Analogously, if X is a set and I^X is the set of all functions on X into I, then we consider I^X as a D-poset in which the partial order and the partial operation $\ominus$ are defined pointwise: $b \le a$ iff $b(x) \le a(x)$ for all $x \in X$ and $a \ominus b$ is defined by $(a \ominus b)(x) = a(x) - b(x)$, $x \in X$. A subset $\mathcal{X} \subseteq I^X$ containing the constant functions 0_X, 1_X and closed with respect to the inherited partial operation "$\ominus$" is a typical D-poset we are interested in; we shall call it a **D-poset of fuzzy sets**.

Clearly, if we identify $A \subseteq X$ and the corresponding characteristic function $\chi_A \in I^X$, then each field $\mathbf{A}$ of subsets of X can be considered as a D-poset $\mathbf{A} \subseteq I^X$ of fuzzy sets: $\mathbf{A}$ is partially ordered ($\chi_B \le \chi_A$ iff $B \subseteq A$) and then $\chi_A \ominus \chi_B$ is defined as $\chi_{A\setminus B}$ provided $B \subseteq A$.

Further, assume that I carries the usual sequential convergence and that I^X and other D-posets of fuzzy sets carry the pointwise sequential convergence. In what follows, we identify I and $I^{\{x\}}$, where $\{x\}$ is a singleton. Let $\mathbf{A}$ be a field of subsets of X considered as a D-poset of fuzzy sets and let p be a probability measure on $\mathbf{A}$. Then p as a map of $\mathbf{A} \subseteq I^X$ into I is **sequentially continuous** and preserves the D-poset structure. For more information concerning the σ-additivity and the sequential continuity of measures see [13].

Denote *ID* the category of all reduced D-posets of fuzzy sets (each two points a, b of the underlying set X are separated by some fuzzy set $u \in \mathcal{X} \subseteq I^X$, i.e. $u(a) \ne u(b)$) carrying the pointwise convergence as objects and the sequentially continuous D-homomorphisms as morphisms. Note that the assumption that all objects of *ID* are reduced plays the same role as the Hausdorff separation axiom T_2: limits are unique and the continuous extensions from dense subobjects are uniquely determined. Observe that since objects of *ID* are subobjects of the power I^X, in proofs of many propositions concerning objects of *ID* we can use the powerfull categorical machinery, e.g., the properties of categorical products. Concerning categorical terminology and constructions the reader is referred to [1].

A typical (classical) object of *ID* is the evaluation of a field of sets. Let $\mathbf{A}$ be a field of sets and let $\mathcal{P}(\mathbf{A})$ be the set of all probability measures on $\mathbf{A}$. For $A \in \mathbf{A}$, put $A^* = \{p(A);\ p \in \mathcal{P}(\mathbf{A})\} \in I^{\mathcal{P}(\mathbf{A})}$ and denote $\mathbf{A}^* = \{A^*;\ A \in \mathbf{A}\}$. Then $\mathbf{A}^*$ is called the **evaluation** of $\mathbf{A}$; observe that $\mathbf{A}^*$ is not anymore a field of sets. Using basic properties of probability measures, it is easy to check that $\mathbf{A}^*$ is a D-poset of fuzzy subsets of $\mathcal{P}(\mathbf{A})$ and that $\mathbf{A}^*$ and $\mathbf{A}$ are isomorphic as D-posets. From the viewpoint of $\mathcal{P}(\mathbf{A})$, $\mathbf{A}^*$ "carries more information" than $\mathbf{A}$. Further details can be found in [15, 24].

Example 4.1 Let $\mathbf{A}$ be a field of subsets of X. If we consider $\mathbf{A}$ as a D-poset (the partial order is defined via inclusion and the difference is defined by $A \ominus B = A \setminus B$ whenever $B \subseteq A$) then $\mathbf{A}$ is a bounded distributive lattice with X and $\emptyset$ as the top and the bottom elements. If h is a D-homomorphism of $\mathbf{A}$ into a field of sets $\mathbf{B}$, then (see Theorem 3.1) h can be considered as a Boolean homomorphism of $\mathbf{A}$ into $\mathbf{B}$. This means that the difference partial operation is "rich enough" and provides all the necessary information about the structure of a field of sets. Now, let p be a D-homomorphism of $\mathbf{A}$ into I. It is straightforward to check that p is an additive probability measure (if p is sequentially continuous, then p is σ-additive). More details can be found in [25] and [17].

Denote *FS* the category having fields of sets as objects and sequentially continuous Boolean homomorphisms as morphisms. It is known that a field $\mathbf{A}$ of subsets of X is a

σ-field iff $\mathbf{A}$ is sequentially closed in $\{0,1\}^X$. Denote *CFS* the full subcategory of *FS* consisting of σ-fields (recall that a subcategory $\mathcal{B}$ of $\mathcal{A}$ is full whenever each $\mathcal{A}$-morphism of a $\mathcal{B}$-object into a $\mathcal{B}$-object is also a $\mathcal{B}$-morphism).

Denote *FSD* the full subcategory of *ID* consisting of fields of sets considered as D-posets and denote *CFSD* its full subcategory consisting of σ-fields.

Proposition 4.2

(i) *The categories FS and FSD are isomorphic.*
(ii) *The categories CFS and CFSD are isomorphic.*

Example 4.3 Let $\mathcal{X} \subseteq I^X$ be a bold algebra of fuzzy sets. If we consider $\mathcal{X}$ as a D-poset, then $\mathcal{X}$ is a bounded distributive lattice with 1_X and 0_X as the top and the bottom elements. Let h be a D-homomorphism of $\mathcal{X}$ into a bold algebra of fuzzy sets $\mathcal{Y} \subseteq I^Y$. In general, h need not be an MV-algebra homomorphism of $\mathcal{X}$ into $\mathcal{Y}$. Indeed, let $\mathbf{B}$ be the σ-field of Lebesgue measurable subsets of [0,1]. Let p be the Lebesgue measure on $\mathbf{B}$. Consider $\mathbb{B}$ and [0,1] as bold algebras and p as a D-homomorphism of $\mathbf{B}$ into [0,1]. For A, B = [0,1/2] we have $\chi_A \oplus \chi_B = \chi_A$ and $p(A) \neq p(A) \oplus p(B) = 1/2 + 1/2 = 1$. This means that in this case the difference partial operation is “not rich enough” and does not provide enough information about the structure of a bold algebra of fuzzy sets. Now, let s be a sequentially continuous D-homomorphism of $\mathcal{X}$ into I. It is known that s is a state. More details can be found in [13].

Denote *BID* the full subcategory of *ID* consisting of bold algebras of fuzzy of subsets considered as D-posets and denote *CBID* its full subcategory consisting of Łukasiewicz tribes.

In the fuzzy probability theory (cf. [3, 4, 17, 18]), the fuzzy events form special bold algebras of fuzzy sets, namely, the set of all measurable functions into I; more details will be given in the next section.

Now we are in a position to give a full answer to Question 2.1.

Answer 4.4 *Traditional probability domains such as fields of sets* (σ*-fields of sets*) *and bold algebras* (*Łukasiewicz tribes*) *can be viewed as full subcategories of ID*, *probability measures on fields of sets and states on bold algebras become morphisms* (*structure preserving maps*).

Since D-posets of fuzzy sets (i.e. objects of *ID*) generalize traditional probability domains, **it is natural to define states on D-posets of fuzzy sets as morphisms into I.**

The category *ID* is cogenerated by I: the objects are subjects of powers I^X. In plain words, in every *ID*-probability domain $\mathcal{X} \subseteq I^X$ (an object of *ID*) “**everything is determined via states**” (morphisms into I):

1. States separate generalized (fuzzy) events;
2. Order, difference, and sequential convergence are categorical (initial with respect to all states, e.g., a sequence $\{u_n\}$ converges to u in $\mathcal{X}$ iff the sequence $\{s(u_n)\}$ converges to $s(u)$ for each state s);
3. The categorical machinery is available and in “chasing diagrams” a key role is played by states.

Let $\mathcal{X} \subseteq I^X$ and $\mathcal{Y} \subseteq I^Y$ be *ID*-objects. Recall that the pairs $(X, \mathcal{X})$, $(Y, \mathcal{Y})$ are called ***ID*-measurable spaces** and a map $f : X \longrightarrow Y$ such that $u \circ f \in \mathcal{X}$ of all $u \in \mathcal{Y}$ is called *ID*-**measurable**. On the one hand, if f is measurable, then $u \mapsto u \circ f$ yields a D-homomorphism $f^{\leftarrow}$ of $\mathcal{Y}$ into $\mathcal{X}$ which is sequentially continuous, hence an *ID*-morphism. On the other hand, if h is an *ID*-morphism of $\mathcal{Y}$ into $\mathcal{X}$ and $\mathcal{Y}$ is **sober** (i.e. each *ID*-morphisms of $\mathcal{Y}$ into I can be represented by a point evaluation at some $y \in Y$), then there exists a unique measurable map f such that $h = f^{\leftarrow}$; this yields a **duality** between *ID*-morphisms and *ID*-measurable maps, i.e., observables and fuzzy random variables in the fuzzy or operational probability developed by in [2, 5, 21].

The "categorical" proofs of the continuity of $f^{\leftarrow}$ and the duality via diagrams (cf. [12, 24]) are rather straightforward and transparent in comparison to the original proofs via functional analysis (cf. [3, 4]).

In the category *ID*, the extension of states from a field of sets $\mathbf{A}$ to the generated σ-field $\sigma(\mathbf{A})$ and from a bold algebra $\mathcal{X} \subseteq I^X$ to the induced Łukasiewicz tribe $\sigma(\mathcal{X}) \subseteq I^X$ become epireflections (see [17, 20]).

5 States in Fuzzy Probability

Consider the following two fundamentally different probability theories: classical Kolmogorovian and fuzzy (the former is called standard and the later is called operational in [3, 4]). Both theories can be "embedded" into the category *ID* in a canonical way. Moreover, they are particular cases of a generalized *ID*-probability theory.

Question 5.1 *What is the categorical background of the transition from classical to fuzzy probability theory*?

Let $(X, \mathbf{A})$, $(Y, \mathbf{B})$ be classical measurable spaces and let $f : X \longrightarrow Y$ be a map. If f is measurable, then the (dual) preimage map $f^d : \mathbf{B} \longrightarrow \mathbf{A}$ defined by $f^d(B) = f^{\leftarrow}(B)$ = $\{x \in X;\ f(x) \in B\}$, $B \in \mathbf{B}$, is a sequentially continuous (the pointwise convergence of characteristic functions) Boolean homomorphism of $\mathbf{B}$ into $\mathbf{A}$. Indeed, the assertion is a corollary of the following straightforward observation.

Observation 5.2 *For each $B \subseteq Y$ we have $\chi_{f^{\leftarrow}(B)} = \chi_B \circ f$ and the measurability of f is equivalent to the following condition*

$$(M) \qquad (\forall B \in \mathbf{B})\,(\exists A \in \mathbf{A})\,[\chi_B \circ f = \chi_A].$$

Observation 5.3 *If p is a probability measure on $\mathbf{A}$, then the composition $p \circ f^d = p_f$ is a probability measure on $\mathbf{B}$. This sends probability measures $\mathcal{P}(\mathbf{A})$ on $\mathbf{A}$ to probability measures $\mathcal{P}(\mathbf{B})$ on $\mathbf{B}$. This way we get basic notions of probability theory*: *random events*, *a random variable f*, *its distribution p_f*, *and the observable f^d*.

In the fuzzy (or operational) probability theory, we start with a map T of $\mathcal{P}(\mathbf{A})$ into $\mathcal{P}(\mathbf{B})$ satisfying a natural measurability condition which guarantees the existence of a dual map T^d of all measurable functions $\mathcal{M}(\mathbf{B})$ of Y into the closed unit interval $I = [0, 1]$ into all measurable functions $\mathcal{M}(\mathbf{A})$ of X into I so that T^d is a sequentially continuous D-homomorphism. This way $\mathcal{M}(\mathbf{A})$ and $\mathcal{M}(\mathbf{B})$ become generalized random events, T becomes a fuzzy random variable, T^d becomes a generalized observable, and each D-homomorphism into I becomes a state. Measurable functions into I can be considered as bold algebras. States on bold algebras are exactly D-homomorphisms into I but, in general, a D-homomorphism need not be an MV-algebra homomorphism. Note that for each Łukasiewicz tribe $\mathcal{X} \subseteq I^X$ there exists a unique σ-field $\mathbf{A}$ of subsets of X such that $\mathcal{X}$ is contained in the Łukasiewicz tribe $\mathcal{M}(\mathbf{A})$ of all measurable functions into I and if $\mathcal{X}$ contains all constant functions $c_X, c \in I$, then $\mathcal{X}$ and $\mathcal{M}(\mathbf{A})$ conicide; such tribes are called **generated** and form a full subcategory *CGBID* of *ID*.

The two probability theories, classical resp. fuzzy, "live in" the full subcategory *CFSD* of *ID* consisting of σ-fields, resp. the full subcategory *CGBID* of *ID* consisting of generated Łukasiewicz tribes (each is of the form of all measurable [0, 1]-valued functions on a σ-field). It was proved in [20] that sending $\mathbf{A}$ to $\mathcal{M}(\mathbf{A})$ yields a functor

$$\mathbf{F} : \mathit{CFSD} \longrightarrow \mathit{CGBID}$$

which is bijective on objects (to each generated Łukasiewicz tribe $\mathcal{X}$ there corresponds a unique σ-field $\mathbf{A}$ such that $\mathbf{F}(\mathbf{A}) = \mathcal{M}(\mathbf{A}) = \mathcal{X}$) but, barred the trivial case, it fails to be bijective on morphisms.

Observation 5.4 *In the Kolmogorovian probability, extending a field of sets* $\mathbf{A}$ *to the generated* σ*-field* $\sigma(\mathbf{A})$ *we pass to a larger probability domain which has the same probability measures and it has some additional properties (e.g. it is* σ*-complete). In the fuzzy probability, extending crisp events* $\mathbf{A} = \sigma(\mathbf{A})$ *to* $\mathcal{M}(\mathbf{A})$ *we pass to a larger probability domain which has "the same states" and it has some additional properties. From the categorical viewpoint, both extensions are epireflections.*

Theorem 5.5 $\mathbf{F}$ *is an epireflection.*

Proof Let $\mathbf{B}$ be a σ-field. First, recall that probability measures on $\mathbf{B}$ are exactly *ID*-morphisms to I. Second, each probability measure p on $\mathbf{B}$ can be uniquely extended (via step functions and the Lebesgue Dominate Convergence Theorem) to an *ID*-morphism $p_{\mathcal{M}}$ (integral) over $\mathcal{M}(\mathbf{B})$. Third, let h be an *ID*-morphism of $\mathbf{B}$ into an object $\mathcal{M}(\mathbf{A})$ of *CGBID*. Since $\mathcal{M}(\mathbf{A})$ is a subobject of powers of I, it follows from the first step and the usual categorical product argument (h followed by a projection of $I^{\mathcal{P}(\mathbf{A})}$ into any factor I is a morphism of $\mathbf{B}$ into the factor in question, hence it can be extended over $\mathcal{M}(\mathbf{B})$ and this implies the existence of a map of $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$ having the desired properties) that h can be uniquely extended to an *ID*-morphism $h_{\mathcal{M}}$ of $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$. □

Theorem 5.6 *Let* $\mathbf{B}$ *be a nontrivial* σ*-field of sets and let* $\mathcal{M}(\mathbf{A})$ *be a generated Łukasiewicz tribe. Then there exist an ID-morphism* h *of* $\mathbf{B}$ *into* $\mathcal{M}(\mathbf{A})$ *and its extension to an ID-morphism* $h_{\mathcal{M}}$ *of* $\mathcal{M}(\mathbf{B})$ *into* $\mathcal{M}(\mathbf{A})$ *such that for each ID-morphism* g *of* $\mathbf{B}$ *into* $\mathbf{A}$ *we have* $h_{\mathcal{M}} \neq \mathbf{F}(g)$.

Proof Let q be a probability measure on $\mathbf{B}$ and let T be the degenerated fuzzy random variable sending each $p \in \mathcal{P}(\mathbf{A})$ to q. Let T^d be the dual observable which maps $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$ and let h be the restriction of T^d to $\mathbf{B}$. It is known (cf. [18]) that h maps each $B \in \mathbf{B}$ (its characteristic function χ_B) to the constant function the value of which is $q(B)$. Assume that q is nontrivial, i.e., there exists $B \in \mathbf{B}$ such that $0 < p(B) < 1$. Then $h(B)$ belongs to $\mathcal{M}(\mathbf{A}) \setminus \mathbf{A}$. The *ID*-morphisms h can be extended to an *ID*-morphism $h_{\mathcal{M}}$ of $\mathcal{M}(\mathbf{B})$ into $\mathcal{M}(\mathbf{A})$, but for each *ID*-morphism g of $\mathbf{B}$ into $\mathbf{A}$ we have $h_{\mathcal{M}} \neq \mathbf{F}(g)$. □

Theorem 5.5 and Theorem 5.6 give at least partial answer to Question 5.1.

Answer 5.7 *Each classical probability domain* (*a σ-field of sets*) *has a unique epireflection into the fuzzy probability domains* (*generated Łukasiewicz tribes*) *so that both domains have "the same" states. However, the transition from classical probability to fuzzy probability is not conservative. Within the later there are observables having quantum character not captured by the classical observables.*

6 Conclusion

As shown in the previous sections, the category *ID* is rich enough to serve as a base category in which both the classical and the fuzzy probability theory can "live in" and fundamental notions such as random events, observables and states become intrinsic. The full subcategories *CFSD* and *CGBID* model the classical and fuzzy probability theory, respectively. The fuzzification, i.e., the transition from classical to fuzzy theory has a meaningful categorical interpretation.

As shown in [18, 19] and [25], even more general theory can be developed within *ID* so that the two theories mentioned above become special cases. Indeed, objects of *ID* can be defined as probability domain, measurable maps as generalized random variables, the dual maps as generalized observables, and *ID*-morphisms into I as generalized states.

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